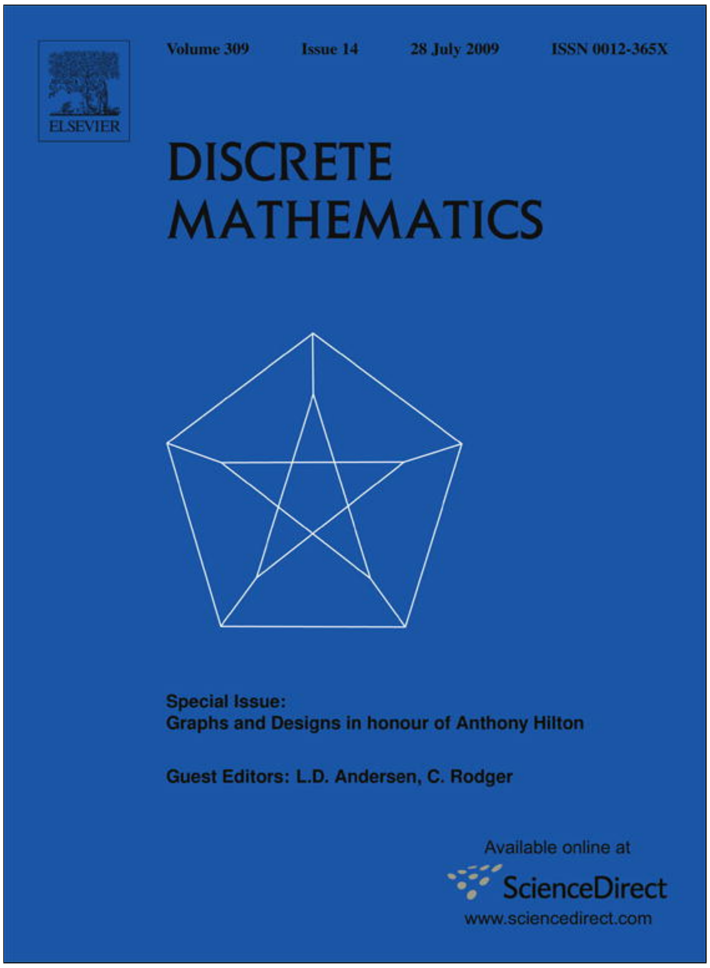
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Ore-type conditions implying 2-factors consisting of short cycles

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ABSTRACT

For every graph G, let $\sigma_2(G) = \min\{d(x) + d(y) \ : \ xy \notin E(G)\}$. The main result of the paper says that every n-vertex graph G with $\sigma_2(G) \geq \frac{4n}{3} - 1$ contains each spanning subgraph H all whose components are isomorphic to graphs in $\{K_1, K_2, C_3, K_4^-, C_5^+\}$. This generalizes the earlier results of Justesen, Enomoto, and Wang, and is a step towards an Ore-type analogue of the Bollobás–Eldridge–Catlin Conjecture.

1. Introduction

Two n-vertex graphs G_1 and G_2 are said to *pack* if there exist injective mappings of their vertex sets onto $[n]$ such that the images of the edge sets do not intersect. In a similar way, one can define the packing of more than two graphs.

The study of extremal problems on packings of graphs was started in the 1970s by Bollobás and Eldridge [3], Sauer and Spencer [18], and Catlin [5].

Sauer and Spencer [18] proved that *two n-vertex graphs pack if the product of their maximum degrees is less than $n/2$*. Kaul and Kostochka [14] characterized the pairs of n-vertex graphs with the product of maximum degrees exactly $n/2$ that do not pack.

The following BEC-conjecture (one of the main conjectures in the area) was posed in 1978 by Bollobás and Eldridge [3], and independently by Catlin [6].

Conjecture 1. *Let G_1 and G_2 be n-vertex graphs with maximum degrees Δ_1 and Δ_2, respectively. If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$, then G_1 and G_2 pack.*

By definition, graphs G_1 and G_2 pack if and only if G_1 contains the complement $\overline{G}_2$ of G_2. In the containment language, the BEC-conjecture states that every n-vertex graph G with minimum degree δ contains each n-vertex graph H such that $(n - \delta)(\Delta(H) + 1) \leq n + 1$.

This conjecture is proved to be true only for some limited classes of graphs, see [1,2,8,4,15]. In particular, Aigner and Brandt [1], and independently Alon and Fisher [2] (for n sufficiently large), proved the special case $\Delta(H) \leq 2$:

Theorem 1. *If G is an n-vertex graph with $\delta(G) \geq (2n - 1)/3$, then G contains each n-vertex graph H with $\Delta(H) \leq 2$.*

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Theorem 1 generalizes an earlier result by Corrádi and Hajnal [7], which says that a $3k$-vertex graph G with minimum degree at least $2k$ contains k disjoint triangles. Another important generalization of the Corrádi–Hajnal result is the Hajnal–Szemerédi Theorem [12] states that *each n-vertex graph G with $\delta(G) \geq (1 - 1/k)n$ contains the graph $H(n, k)$ whose every component is K_k, given that k divides n*. This theorem is the partial case of the BEC-conjecture for G_2 being the disjoint union of complete graphs of the same size.

The above-mentioned results have the spirit of Dirac's Theorem [9] (which says that every n-vertex graph with minimum degree at least $n/2$ contains a hamiltonian cycle) in the sense that these results guarantee the existence of some subgraph if the minimum degree of the graph is large enough. Ore [17] gave a different sufficient condition for hamiltonicity: he proved that every n-vertex graph G with

$$\sigma_2(G) = \min_{xy \notin E(G)} \{\deg(x) + \deg(y)\} \geq n$$

contains a hamiltonian cycle. Justesen [13] proved an Ore-type version of the Corrádi–Hajnal Theorem by showing that every n-vertex graph G with $\sigma_2(G) \geq 4n/3$ contains $\lfloor n/3 \rfloor$ disjoint triangles. Enomoto [10], and Wang [19] sharpened this result. In particular, they proved the following.

Theorem 2. *For each positive integer k, every $3k$-vertex graph G with $\sigma_2(G) \geq 4k - 1$ contains k disjoint triangles.*

Our main result is the following.

Theorem 3. *Each n-vertex graph G with*

$$\sigma_2(G) \geq \frac{4n}{3} - 1 \tag{1}$$

contains all spanning subgraphs whose components are isomorphic to graphs in $\mathcal{H} = \{K_1, K_2, C_3, K_4^-, C_5^+\}$.

Here C_5^+ denotes a cycle of length five with a chord. Note that K_4^- can also be considered as C_4^+, i.e. a cycle of length four with a chord.

Condition (1) cannot be weakened. For example, for each integer $k \geq 2$, let $G(k)$ denote the complement of the disjoint union $K_k \cup K_k \cup K_{k-2}$. Its number of vertices, $n(k)$, is $3k - 2$ and $\sigma_2(G(k)) = 4k - 4 = \frac{4n(k)-1}{3} - 1$ which is just 1/3 less than the lower bound in (1). However, $G(k)$ does not contain the graph $H(k)$ which is the disjoint union of $k - 1$ triangles and a single vertex. It particular, Theorem 3 generalizes and extends the above-mentioned results of Justesen, Enomoto and Wang.

Theorem 3 is also a step towards an Ore-type analogue of the BEC-conjecture. We state and discuss this analogue in the next section in terms of graph packing. In Section 3 we present some technical results on the existence of some subgraphs in dense graphs on at most 12 vertices. The proofs in this section can be omitted at first reading. In Section 4 we prove the following weakening of Theorem 3.

Theorem 4. *Each n-vertex graph G with*

$$\sigma_2(G) \geq \frac{4n}{3} - 1$$

contains all spanning subgraphs whose components are isomorphic to graphs in $\mathcal{H}_1 = \{K_1, K_2, C_3, K_4^-\}$.

In Section 5 we prove two auxiliary statements, and in the final section we prove the main result. The idea of the proofs of most results below is as follows. We have a graph G satisfying (1) and a graph H that we want to show to be embeddable into G. We also know that G contains another graph H' that is obtained from H by replacing one (small) component, say F, with a bit 'simpler' component F'. Using (1), we show that there is some embedding $f : V(H') \to V(G)$ of H' into G such that there are 'many' edges in G between $f(V(F'))$ and the image of some other component F'' of H'. Then we prove that under these conditions $G[f(V(F')) \cup f(V(F''))]$ contains vertex-disjoint copies of F and F''.

The notation used is mostly from [20]. Let G be a graph. For $W, U \subseteq V(G)$, $e(W, U)$ is the number of edges connecting W with U. For $W \subseteq V(G)$ and $x \in V(G)$, $N_W(x)$ is the set of neighbors of x in W and $d_W(x) = |N_W(x)|$. Also, $G[x_1, \ldots, x_k]$ (respectively, $G[W - x_1 - \cdots - x_k + y_1 + \cdots + y_l]$) denotes the subgraph of G induced by the set $\{x_1, \ldots, x_k\}$ (respectively, by the set $W \cup \{y_1, \ldots, y_k\} \setminus \{x_1, \ldots, x_k\}$).

2. A graph packing conjecture

As mentioned in the introduction, a graph G contains a graph H if and only if H packs with the complement $\overline{G}$ of G. Ore-type conditions look more natural for packing graphs than for embedding graphs. Indeed, let $\theta(G) = \max_{xy \in E(G)} \{\deg(x) + \deg(y)\}$. In terms of θ, Ore's Theorem claims that every n-vertex graph G with $\theta(G) \leq n - 2$ packs with the cycle C_n of length n. Note that $\theta(G) = \Delta(L(G)) + 2$, where $L(G)$ is the line graph of G. By definition, for every graph G,

$$\Delta(G) + \delta(G) \leq \theta(G) \leq 2\Delta(G). \tag{2}$$

In [16] Dirac-type packing results of Sauer and Spencer [18] and Kaul and Kostochka [14] mentioned above were extended to the following Ore-type result.

Theorem 5. *If two n-vertex graphs G_1 and G_2 satisfy $\theta(G_1)\Delta(G_2) \le n$, then G_1 and G_2 pack, with the following exceptions:*
(I) *G_1 is a perfect matching and G_2 is either $K_{n/2,n/2}$ with $n/2$ odd or contains $K_{n/2+1}$;*
(II) *G_2 is a perfect matching and G_1 is either $K_{r,n-r}$ with r odd or contains $K_{n/2+1}$.*

In [15] we posed the following conjecture which by (2) extends the BEC-Conjecture.

Conjecture 2. *If G_1 and G_2 are n-vertex graphs and $(0.5\theta(G_1) + 1)(\Delta(G_2) + 1) \le n + 1$, then G_1 and G_2 pack.*

Theorem 3 implies the partial case of Conjecture 2 when every component of G_2 is a cycle of length at most five or a short path.

Remark. One of the referees suggested to consider an Ore-type analogue of the result by Fan and Kierstead [11] that every n-vertex graph G with $\delta(G) \ge (2n - 1)/3$ contains the square of a hamiltonian path. That would be a challenging problem.

3. On small dense graphs with a C_4-subgraph

In this section we present some technical facts on the existence of K_4^--subgraphs in small (on at most 12 vertices) dense graphs. The reader can skip it at first reading.

Lemma 1. *Let V_1 and V_2 be disjoint vertex subsets of a graph F such that $F_1 = F(V_1) = K_3$, $F_2 = F(V_2)$ is the 4-cycle $y_1y_2y_3y_4$ and $e(V_1, V_2) \ge 9$. If each vertex in V_2 is adjacent to some vertex in F_1, then $V_1 \cup V_2$ can be partitioned into two sets V_1' and V_2' such that $F(V_1')$ is K_3 and $F(V_2')$ contains K_4^-.*

Proof. Let $V_1 = \{x_1, x_2, x_3\}$. First suppose that some x_i is adjacent to every vertex in V_2. Some vertex y_j is adjacent to at least $\lceil 9/4\rceil = 3$ vertices in V_1. Then $F(V_1 - x_i + y_j) = K_3$ and $F(V_2 - y_j + x_i) = K_4^-$.

The only other possibility is that each vertex in V_1 has exactly 3 neighbors in V_2. Suppose that $N_{V_2}(x_1) = V_2 - y_4$. If both x_2 and x_3 are neighbors of y_4, then we let $V_1' = V_1 - x_1 + y_4$ and $V_2' = V_2 - y_4 + x_1$. So, we can assume that $N_{V_2}(x_2) = V_2 - y_4$. Then under conditions of the lemma, $x_3y_4 \in E(F)$. Vertex x_3 must also be adjacent to some $y \in \{y_1, y_3\}$, say to y_1. Then we let $V_1' = \{x_3, y_1, y_4\}$ and $V_2' = \{x_1, x_2, y_2, y_3\}$. □

Lemma 2. *Let V_1 and V_2 be disjoint vertex subsets of a graph F such that*
(a) *$V_1 = \{x_1, x_2, x_3, x_4\}$, and $F_1 = F(V_1) = K_4$;*
(b) *$F_2 = F(V_2)$ is the 4-cycle $y_1y_2y_3y_4$;*
(c) *$|E_F(V_1, V_2)| \ge 11$.*
If $F(V_1 \cup V_2)$ does not contain two vertex-disjoint copies of K_4^-, then there are $x_4 \in V_1$ and $y_4 \in V_2$ such that
(i) *$N_{F_2-y_4}(x_1) = N_{F_2-y_4}(x_2) = N_{F_2-y_4}(x_3) = \{y_1, y_2, y_3\}$;*
(ii) *y_4 has at most one neighbor in V_1 and $|E_F(V_1, V_2)| = 11$.*

Proof. Assume that V_1 and V_2 satisfy conditions (a)–(c), but $F(V_1 \cup V_2)$ does not contain two vertex-disjoint copies of K_4^-. First we prove that

$$d_{F_2}(x_i) \le 3 \quad \text{for each } i. \tag{3}$$

Indeed, if $d_{F_2}(x_i) = 4$, then we can choose some $y \in V_2$ with $d_{F_1}(y) \ge 3$ and let $F_1' = F(V_1 - x_i + y)$ and $F_2' = F(V_2 + x_i - y)$.

Assume that y_4 has the fewest neighbors in V_1.

CASE 1: $d_{V_1}(y_4) = 0$. By (c), at least 3 vertices in V_1 have three neighbors in V_2, each. Thus in this case both (i) and (ii) hold.

CASE 2: $d_{V_1}(y_4) = 1$. Suppose $x_4y_4 \in E(F)$. By (c), every $y \in V_2 - y_4$ has $d_{V_1}(y) \ge 2$. Hence if $d_{V_2}(x_4) = 3$, then we get two vertex-disjoint copies of K_4^- by switching x_4 with its non-neighbor $y \in V_2$. If $d_{V_2}(x_4) \le 2$, by (c), each of x_1, x_2, x_3 has exactly 3 neighbors in V_2, i.e. (i) holds. Also there must be equality in (c), so (ii) also holds.

CASE 3: $d_{V_1}(y_4) \ge 2$. By (c) and (3), some $x \in V_1$ has exactly 3 neighbors in V_2. Then switching x with its non-neighbor y in V_2, we obtain two vertex-disjoint copies of K_4^-, a contradiction. This finishes the proof. □

Lemma 3. *Let V_1 and V_2 be disjoint vertex subsets of a graph F such that*
(a) *$V_1 = \{x_1, x_2, x_3, x_4\}$, and $F_1 = F(V_1) = K_4^-$ with $x_1x_4 \notin E(F)$;*
(b) *$F_2 = F(V_2)$ is the 4-cycle $y_1y_2y_3y_4$;*
(c) *$|E_F(V_1, V_2)| \ge 11$.*
If $F(V_1 \cup V_2)$ does not contain two vertex-disjoint copies of K_4^-, then either $F(V_1 \cup V_2)$ contains a copy of K_4 and disjoint from it copy of C_4, or there are $x \in \{x_2, x_3\}$ and $y_4 \in V_2$ such that
(i) *$N_{F_2-y_4}(x') = \{y_1, y_2, y_3\}$ for each $x' \in V_1 - x$;*
(ii) *y_4 and y_2 have no common neighbors in V_1.*

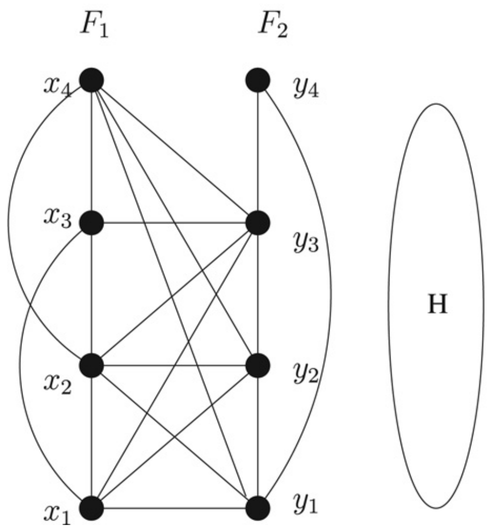

Fig. 1.

Proof. Assume that y_4 has the fewest neighbors in V_1.

CASE 1: $d_{V_1}(y_4) = 0$ (see Fig. 1). By (c), at least 3 vertices in V_1 have three neighbors in V_2, each. So, if (i) does not hold, then we may assume that x_4 has exactly two neighbors in $V_2 - y_4$, and every other $x \in V_1$ is adjacent to all y_1, y_2, and y_3. If $x_4y_2 \in E(F)$, then switching y_2 with x_1 we obtain a copy of K_4 and a disjoint from it 4-cycle. Otherwise, $x_4y_1, x_4y_3 \in E(F)$ and we get a copy of K_4 and a disjoint from it 4-cycle by switching y_2 with x_4.

CASE 2: $d_{V_1}(y_4) = 1$. Suppose $x_iy_4 \in E(F)$.

Subcase 2.1: $d_{V_2}(x_i) \geq 3$. Define y to be the non-neighbor of x_i in V_2 if $d_{V_2}(x_i) = 3$ or any $y \in V_2$ with 4 neighbors in V_1 otherwise. Note that such y exists by (c) when $d_{V_1}(y_4) = 1$. We try to switch x_i with y. We do not get two disjoint copies of K_4^- or a copy of K_4 and a copy of C_4 only if $d_{V_2}(x_i) = 3$, y has exactly two neighbors in V_1 and $F_1 - x_i$ is not a K_3. In particular, we may assume that $i = 2$. Also from (c) we conclude that $d_{V_1}(y') = 4$ for both $y' \in V_2 - y_4 - y$. By the symmetry between y_1 and y_3, we may assume that $y_3 \in V_2 - y_4 - y$. By the symmetry between x_1 and x_4, we may assume that $x_1y \in E(F)$. Then either of $F[y_3, y_4, x_2, x_4]$ and $F[y_1, y_2, x_1, x_3]$ contains K_4^-.

Subcase 2.2: $d_{V_2}(x_i) \leq 2$. By (c), $d_{V_2}(x_i) = 2$ and each $x \in V_1 - x_i$ is adjacent to y_1, y_2, and y_3. Thus (i) holds, unless $i \in \{1, 4\}$. Suppose, $i = 4$. Then we have $F[x_1, x_2, y_1, y_2] = K_4$ and the 4-cycle (x_3, y_3, y_4, x_4). So, we only need to prove (ii) in the case $i \in \{2, 3\}$. Suppose that $i = 3$ and $x_3y_2 \in E(F)$. Then $F[V_1 - x_1 + y_2] = K_4$ and $F[V_2 - y_2 + x_1] = C_4$. This proves (ii).

CASE 3: $d_{V_1}(y_4) \geq 2$.

Subcase 3.1: $d_{V_2}(x_1) \geq 3$. Switch x_1 with its non-neighbor y in V_2, if $d_{V_2}(x_1) = 3$, and with any $y \in V_2$, otherwise. In both cases we obtain two K_4^-.

By the symmetry between x_1 and x_4, the remaining case is the following.

Subcase 3.2: $d_{V_2}(x_1) \leq 2$ and $d_{V_2}(x_4) \leq 2$. By (c), we can assume that $d_{V_2}(x_2) = 4$. If $d_{V_1}(y) = 4$ for some $y \in V_2$, then we switch x_2 with y and get two K_4^-. Otherwise, we can assume that

$$d_{V_1}(y_1) = d_{V_1}(y_2) = d_{V_1}(y_3) = 3. \tag{4}$$

By the symmetry between x_1 and x_4, we can assume that $x_1y_3 \in E(F)$. Then $F[x_1, x_2, y_3, y_4]$ has at least five edges. If $F[y_1, y_2, x_3, x_4]$ also has at least five edges, then we are done. Otherwise, by (4) both y_1 and y_2 are adjacent to x_1, a contradiction to $d_{V_2}(x_1) \leq 2$. □

Lemma 4. *Let V_1, V_2, and V_3 be disjoint vertex subsets of a graph F such that*

(a) *$V_1 = \{x_1, x_2, x_3, x_4\}$, and $F_1 = F(V_1) \supset K_4^-$ with x_1x_4 possibly not in $E(F)$;*

(b) *$F_2 = F(V_2)$ is the 4-cycle $y_1y_2y_3y_4$;*

(c) *$F_3 = F(V_3) \in \{K_1, K_2, C_3, K_4, K_4^-\}$;*

(d) *$N_{F_2}(x_1) = N_{F_2}(x_2) = N_{F_2}(x_4) = \{y_1, y_2, y_3\}$.*

If

$$d_{F_3}(y_1) + d_{F_3}(y_3) + 2(d_{F_3}(y_2) + d_{F_3}(y_4)) > 4|V_3|, \tag{5}$$

then $V_1 \cup V_2 \cup V_3$ can be partitioned into sets V_1', V_2' and V_3' so that $F(V_1')$ and $F(V_2')$ contain K_4^- and $F(V_3')$ contains F_3.

Proof. By the symmetry between y_1 and y_3, we will assume that $d_{F_3}(y_1) \geq d_{F_3}(y_3)$.

If $F_3 = K_1$ with $V(F_3) = \{u\}$, then $y_1, y_2, y_4 \in N(u)$. Then we have $F_1, F' = \{y_3\}$ and $K_4^- \subseteq G[u, y_1, y_2, y_4]$.

Suppose $F_3 = K_2$ with $V(F_3) = \{u_1, u_2\}$. By (5), either $F[u_1, u_2, y_1, y_2]$ or $F[u_1, u_2, y_3, y_4]$ has at least 5 edges. If it is $F[u_1, u_2, y_1, y_2]$, then we have F_1, F' with $V(F') = \{y_3, y_4\}$ and $K_4^- \subseteq F[u_1, u_2, y_1, y_2]$. The other possibility is very similar.

If F_3 is 3-cycle (z_1, z_2, z_3), then $d_{F_3}(y_2) + d_{F_3}(y_4) \geq 4$. If $d_{F_3}(y_4) \geq 2$, then $G[y_4, V(F_3)] \supseteq K_4^-$, $G[y_3, x_2, x_3, x_4] = F_1'$, and $G[x_1, y_1, y_2] = K_3$. Suppose now that $d_{F_3}(y_4) \leq 1$. Then $d_{F_3}(y_2) = 3$ and $d_{F_3}(y_4) = 1$. It follows that $d_{F_3}(y_1) = 3$ and $d_{F_3}(y_3) \geq 2$. We may assume that $y_4z_1 \in E(G)$ and $y_3z_2 \in E(G)$. Then $F_2 \cup F_3$ can be decomposed into 3-cycle $G[z_2, y_2, y_3]$ and $G[y_1, y_4, z_1, z_3] \supseteq K_4^-$.

The last case is that F_3 is a 4-cycle (with at least one chord) (z_1, z_2, z_3, z_4). Then since $e(\{y_1, y_3\}, F_3) \leq 8$, $d_{F_3}(y_2) + d_{F_3}(y_4) \geq 5$.

If $d_{F_3}(y_2) + d_{F_3}(y_4) = 5$, then $d_{F_3}(y_1) + d_{F_3}(y_3) \geq 7$, and hence $d_{F_3}(y_1) = 4$ and $d_{F_3}(y_3) \geq 3$. Let $z \in F_3$ be a common neighbor of y_2 and y_4. If $zy_3 \notin E(G)$, then we have three K_4^-'s: $G[V(F_3) - z + y_3]$, $G[y_1, y_2, z, y_4]$ and F_1. On the other hand, if $zy_3 \in E(G)$, we also have three K_4^-'s: $G[V(F_3) - z + y_1]$, $G[y_2, y_3, y_4, z]$ and F_1.

The following claim will be helpful.

Claim 1. *Suppose that some $y \in \{y_1, y_3\}$ has a common neighbor, say $z_1 \in F_3$, with y_2. Then y_4 has at most two neighbors in $\{z_2, z_3, z_4\}$, and if it has exactly two neighbors in $\{z_2, z_3, z_4\}$, then $z_2z_4 \notin E(F_3)$ (and hence $z_1z_3 \in E(F_3)$).*

Proof. Let $y' \in \{y_1, y_3\} - y$. If y_4 is adjacent to each of z_2, z_3, z_4, or is adjacent to two of them and $z_2z_4 \in E(F_3)$, then we have three K_4^-'s: $F[V(F_3) - z_1 + y_4]$, $F[x_1, y, y_2, z_1]$ and $F[y', x_2, x_3, x_4]$. □

If $d_{F_3}(y_2) + d_{F_3}(y_4) = 8$, then $d_{F_3}(y_1) \geq 1$. This contradicts Claim 1.

Suppose that $d_{F_3}(y_2) + d_{F_3}(y_4) = 7$. In this case, $d_{F_3}(y_1) + d_{F_3}(y_3) \geq 3$, and therefore $d_{F_3}(y_1) \geq 2$. Hence y_2 and y_1 have a common neighbor, say, z_1 in F_3. In view of Claim 1, since $d_{F_3}(y_2) + d_{F_3}(y_4) = 7$, we can assume that $d_{F_3}(y_2) = 4$ and $z_2z_4 \notin E(F_3)$. Then $z_1z_3 \in E(F_3)$ and $N_{F_3}(y_1) = \{z_1, z_3\}$. Furthermore, by symmetry, we may assume that $y_3z_1 \in E(F)$ and that $y_4z_2 \in E(F)$. In this case, we replace F_2 and F_3 by $F_2 - y_2 + z_1$ and $F_3 - z_1 + y_2$.

Finally, suppose that $d_{F_3}(y_2) + d_{F_3}(y_4) = 6$. Then $d_{F_3}(y_1) + d_{F_3}(y_3) \geq 5$. By Lemma 2 if $F(V_2 \cup V_3)$ does not contain two vertex-disjoint copies of K_4^-, then $F_3 \neq K_4$. By Lemma 3, if $F(V_2 \cup V_3)$ does not contain two vertex-disjoint copies of K_4^- or a copy of K_4 and a copy of C_4, then there is a vertex, say z_4, in V_3 and some $y_i \in V_2$ such that each $y \in V_2 - y_i$ is adjacent to each $z \in V_3 - z_4$ and y_i has at most one neighbor in V_3. Since $d_{F_3}(y_2) + d_{F_3}(y_4) = 6$, $i = 3$. Furthermore, in this case $z_4z_2 \in E(F)$. Then we have the following 3 copies of K_4^-: $F[x_2, x_3, x_4, y_3]$, $F[x_1, y_1, y_2, z_1]$, and $F[y_4, z_2, z_3, z_4]$. □

The next lemma is similar.

Lemma 5. *Let V_1, V_2, and V_3 be disjoint vertex subsets of a graph F such that*

(a) $V_1 = \{x_1, x_2, x_3\}$, *and* $F_1 = F(V_1) = K_3$;
(b) $F_2 = F(V_2)$ *is the 4-cycle* $y_1y_2y_3y_4$;
(c) $F_3 = F(V_3) \in \{K_1, K_2, C_3, K_4, K_4^-\}$;
(d) $N_{F_2}(x_1) = N_{F_2}(x_2) = N_{F_2}(x_3) = \{y_1, y_2, y_3\}$.

If

$$d_{F_3}(y_1) + d_{F_3}(y_3) + 2(d_{F_3}(y_2) + d_{F_3}(y_4)) > 4|V_3|, \tag{6}$$

then $V_1 \cup V_2 \cup V_3$ can be partitioned into sets V_1', V_2' and V_3' so that $F(V_1')$ is K_3, $F(V_2')$ contains K_4^-, and $F(V_3')$ contains F_3.

The proof of this lemma mimics that of Lemma 4 but is much simpler, so we omit it.

4. Packing 3- and 4-cycles

In this section, we will prove Theorem 4. Let $\mathcal{H}_n$ be the class of n-vertex graphs whose every component is either K_1, or K_2, or K_3, or K_4^-. Let $\mathcal{H}_n'$ consist of graphs H in $\mathcal{H}_n$ such that at most one component of H is K_2.

It is enough to prove the theorem for graphs in $\mathcal{H}_n'$, since each graph $H \in \mathcal{H}_n$ is contained in a graph $H' \in \mathcal{H}_n'$ (we can replace two copies of K_2 in H by a copy of K_4^-). Let G satisfy the conditions of the theorem. Suppose, for a contradiction, that G does not contain some graph in $\mathcal{H}_n'$. Among such 'bad' graphs in $\mathcal{H}_n'$ choose a graph H_0 with fewest components that are K_4^-s. Suppose that H_0 has no K_4^--components. The following corollary of Theorem 2 handles this case.

Proposition 1. *Let $\mathcal{H}_n''$ be the class of n-vertex graphs whose every component is either K_1, or K_2, or K_3, and at most one of these components is K_2. Then every n-vertex graph G with $\sigma_2(G) \geq 4n/3 - 1$ contains each graph in $\mathcal{H}_n''$.*

Proof. If $n = 3k$, then the statement directly follows from Theorem 2.

If $n = 3k + 1$, then $\sigma_2(G) \geq \lceil 4(3k + 1)/3 - 1 \rceil = 4k + 1$ and hence for any vertex $v \in V(G)$, graph $G - v$ satisfies the conditions of Theorem 2. Hence $G - v$ contains each graph in $\mathcal{H}_{n-1}''$. On the other hand, if $n = 3k + 1$, then at least one component of any graph $H \in \mathcal{H}_n''$ is K_1. This settles the case $n = 3k + 1$.

If $n = 3k - 1$, then $\sigma_2(G) \geq \lceil 4(3k - 1)/3 - 1 \rceil = 4k - 2$. Adding to G a new vertex z adjacent to each other vertex, we get a graph G^* satisfying Theorem 2. Hence G^* contains k disjoint triangles. It follows that G contains the graph H_n^* that has one K_2-component and $k - 1$ K_3-components. But such an H_n^* contains each graph in $\mathcal{H}_n''$. □

Assume now that H_0 contains some K_4^-s.

Proposition 2. *Let H_0'' be obtained from H_0 by replacing one component K_4^- with C_4. Then G contains H_0''.*

Proof. Suppose not. Let H_0' be the graph obtained from H_0 by replacing one component K_4^- with the graph $C_3 \cup K_1$. By the choice of H_0, G contains H_0'. Among all copies of H_0' contained in G choose a copy H with most components K_4^- embedded into K_4-subgraphs of G.

Choose in H a K_3-component with vertex set $W = \{w_1, w_2, w_3\}$ and a K_1-component v. By the choice of H, $K_4^- \not\subseteq G[w_1, w_2, w_3, v]$. Then v has at most one neighbor in W.

For every $U \subseteq V(G)$, define $D(U) = 3d_U(v) + d_U(w_1) + d_U(w_2) + d_U(w_3)$.

CASE 1. $D(V(G)) \geq 3\sigma_2(G)$. In this case,

$$D(V(G)) \geq 3\sigma_2(G) - 7 - 3 \geq 4n - 3 - 10 > 4(n-4),$$

and hence there is a component of H with vertex set $U \subset V(G)$ such that

$$D(U) > 4|U|. \tag{7}$$

If $U = \{u\}$, then u has at least two neighbors in W and thus $G[W+u]$ contains K_4^-. But then G contains H_0, a contradiction.

Suppose that $U = \{u_1, u_2\}$ and $G[U] = K_2$. By (7), v has a neighbor, say, u_1 in U. If u_2 has at least two neighbors in W, then $G[W + u_2] \supseteq K_4^-$ and $G[v, u_1] = K_2$, a contradiction to the choice of G. Otherwise, again by (7), $vu_2 \in E(G)$. Then similarly, u_1 also has at most one neighbor in W, a contradiction to (7).

If $G[U]$ is a triangle, then $e(W, U) \leq 9$, and hence there are at least two edges between v and U. Thus $G[U + v]$ contains K_4^- and $G[W]$ contains a 3-cycle. Again G contains H_0, a contradiction.

Now suppose that $U = \{u_1, u_2, u_3, u_4\}$ and $G[U] \supseteq K_4^-$ with possible non-edge u_1u_3. By (7), $3d_U(v) + e(W, U) \geq 17$. Since $e(W, U) \leq 12$, $d_U(v) \geq 2$. If $d_U(v) = 2$, then u_1 or u_3 (we may assume u_1) has three neighbors in W, thus $G[W + u_1] = K_4$ and then by the choice of H, $G[U]$ is also K_4. Note that if some non-neighbor of v in U has at least two neighbors in W, then we again can embed H_0 into G. Thus $16 < D(U) \leq 3d_U(v) + 4 - d_U(v) + 3d_U(v) = 4 + 5d_U(v)$, and therefore $d_U(v) \geq 3$, a contradiction. So $d_U(v) \geq 3$. If $u_2, u_4 \in N(v)$, then $G[U] = K_4$, since $G[N_U[v]]$ contains K_4. But then as above each vertex in U has at most one neighbor in W, and we have $16 < D(U) \leq 3d_U(v) + 4 \leq 16$, a contradiction. So we assume that $u_1, u_2, u_3 \in N(v)$ and $u_4 \notin N(v)$. Then again, every vertex in U has at most one neighbor in W, and we have a contradiction.

CASE 2. $D(V(G)) < 3\sigma_2(G)$. Then v has exactly one neighbor in W, say w_1. By the definition of σ_2, $d(v) + d(w_i) \geq \sigma_2(G)$ for $i = 2, 3$, and hence $2d_{G-W-v}(v) + d_{G-W-v}(w_2) + d_{G-W-v}(w_3) \geq 2\sigma_2(G) - 6 > \frac{8}{3}(n-4)$. Therefore there is a component of H with vertex set $U \subset V(G)$ such that

$$2d_U(v) + d_U(w_2) + d_U(w_3) > \frac{8}{3}|U|. \tag{8}$$

On the other hand, since $D(V(G)) < 3\sigma_2(G)$, we have $d(v) + d(w_1) < \sigma_2(G)$ and hence

$$d(w_1) < \min\{d(w_2), d(w_3)\}. \tag{9}$$

If $U = \{u\}$, then u is adjacent to v and to at least one of w_2 and w_3. Thus we have a 4-cycle, a contradiction.

If $U = \{u_1, u_2\}$ and $G[U] = K_2$, then by (8), $2d_U(v) + d_U(w_2) + d_U(w_3) \geq 6$. Similarly to Case 1, if u_i is adjacent to both w_2 and w_3, and $u_{3-i}v \in E(G)$, then we have disjoint K_4^- and K_2, a contradiction. Hence, $d_{W-w_1}(u_i) + 2d_{\{v\}}(u_{3-i}) \leq 3$ for $i = 1, 2$. It is possible only if $vu_1, vu_2 \in E(G)$ and each u_i has exactly one neighbor in $\{w_2, w_3\}$. If this is the same neighbor, say w_2, then we have $G[v, u_1, u_2, w_2] = K_4^-$ and $G[w_3, w_1] = K_2$. If these neighbors are distinct, say $u_1w_2, u_2w_3 \in E(G)$, then $G[v, u_1, w_2, w_1] = C_4$ and $G[w_3, u_2] = K_2$. Both outcomes contradict the choice of G.

If $G[U] = K_3$, then $2d_U(v) \geq 9 - 6 = 3$ and so v is adjacent to at least two of the vertices in U. Hence $G[U + v]$ contains K_4^- and $G[W]$ contains a 3-cycle, a contradiction.

If $G[U] = K_4$, then by (8), $2d_U(v) \geq 11 - 8 = 3$ and so $d_U(v) \geq 2$. If there is $u \in V(U)$ such that $d_W(u) \geq 2$ and $d_{U-u}(v) \geq 2$, then we partition $G[W \cup U + v]$ into two K_4^-: $G[W + u]$ and $G[U - u + v]$, a contradiction to the choice of G. Thus the only possibility to satisfy (8) is that $d_U(v) = 4$ and each vertex in U has at most one neighbor in W. Since by (8), $d_U(w_2) + d_U(w_3) \geq 11 - 8 = 3$, we may assume that for some $u \in U$, $uw_3 \in E(G)$ and therefore $uw_1, uw_2 \notin E(G)$. By (9), $d(u) + d(w_3) \geq d(u) + d(w_1) \geq \sigma_2(G)$, and hence $3d(u) + d(w_1) + d(w_2) + d(w_3) \geq 3\sigma_2$. Since $G[U - u_1 + v] = K_4$, we come to Case 1, which is resolved.

Finally suppose that $U = \{u_1, u_2, u_3, u_4\}$ and $G[U] = K_4^-$ with non-edge u_2u_4. As in the previous paragraph, $d_U(v) \geq 2$. Suppose first that $N(v) \cap U = \{u_i, u_{i+1}\}$ for some i, say, for $i = 1$. Then by (8), $d_U(w_2) + d_U(w_3) \geq 7$. In particular, $d_{U-u_2}(w_j) \geq 2$ for $j = 2, 3$ and we may assume that $w_3u_2 \in E(G)$. Then $G[U - u_2 + w_2]$ contains K_4^- and $G[W - w_2 + v + u_2]$ contains C_4. If $N(v) \cap U \neq \{u_i, u_{i+1}\}$ for some i, then $N(v) \cap U \supseteq \{u_i, u_{i+2}\}$ for some $i \in \{1, 2\}$. If for some $j \neq i, i+2$, vertex u_j has at least two neighbors in W, then $G[W + u_j]$ contains K_4^- and $G[U - u_j + v]$ contains C_4. Hence for each $j \neq i, i+2$, vertex u_j has at most one neighbor in W. In particular, by (8), $d_U(v) \geq 3$. If $N_U(v)$ contains a triangle, say $U - u_4$, then $G[U + v]$ contains $K_4 \cup K_1$, a contradiction to the choice of H. Otherwise, we may assume that $N_U(v) = \{u_2, u_3, u_4\}$. In this case, each $u \in U$ has at most one neighbor in W, which together with (8) yields $d_U(v) \geq 4$. A contradiction to our last assumption finishes the proof of Proposition 2. □

Fix an embedding of H_0'' into G provided by Proposition 2. Suppose that the C_4-component of H_0'' is embedded into 4-cycle (w_1, w_2, w_3, w_4) in G. Let $W = \{w_1, w_2, w_3, w_4\}$. By the choice of G, $w_1w_3, w_2w_4 \notin E(G)$. Since $(d(w_1)+d(w_3))+(d(w_2)+d(w_4)) \geq 2\sigma_2(G)$, we have $\sum_{i=1}^{4} d_{G-W}(w_i) \geq 2\sigma_2(G) - 8 \geq \frac{8n}{3} - 10 > \frac{8}{3}(n-4)$. Hence there exists a component of H_0'' mapped to a set $U \subset V(G)$ with

$$e(W, U) > \frac{8}{3}|U|. \tag{10}$$

CASE 1. $U = \{v\}$. Since $e(v, W) \geq 3$, $N(v) + v$ contains K_4^- and $G[W - N(v)]$ is K_1. This contradicts the choice of G.

CASE 2. $U = \{u_1, u_2\}$ and $G[U] = K_2$. Since $e(W, U) \geq \lceil 16/3 \rceil = 6$, we may assume that $e(\{w_1, w_2\}, U) \geq 3$. Then $G[w_1, w_2, u_1, u_2]$ contains K_4^- and $G[w_3, w_4] = K_2$, a contradiction to the choice of G.

CASE 3. $U = \{u_1, u_2, u_3\}$ and $G[U] = K_3$. Then $e(W, U) \geq 9$. Suppose that we cannot decompose $G[U \cup W]$ into K_4^- and K_3. Then by Lemma 1, there is a vertex, say w_4, in W such that $N_W(u_i) = W - w_4$ for $i = 1, 2, 3$. Since $d_{G-W}(w_1) + d_{G-W}(w_3) + 2(d_{G-W}(w_2) + d_{G-W}(w_4)) \geq 3\sigma_2(G) - 8 > 4(n-4)$, there exists a component of H_0'' mapped to a set $U' \subset V(G)$ with $e(U', W) = d_{U'}(w_1) + d_{U'}(w_3) + 2(d_{U'}(w_2) + d_{U'}(w_4)) > 4|U'|$. Since U does not satisfy this condition, $U' \neq U$. Applying Lemma 5 with $F_1 = U$, $F_2 = U_0$ and $F_3 = U'$, we again get a contradiction to the choice of G.

CASE 4. $U = \{u_1, u_2, u_3, u_4\}$ and $G[U] = K_4^-$ with $u_1u_4 \notin E(G)$. Suppose that we cannot partition $G[U \cup W]$ into two K_4^- or into K_4 and C_4. Since $e(W, U) \geq 11$, by Lemma 3, we may assume that each of w_1, w_2, and w_3 is adjacent to each of u_1, u_2, and u_4 and that w_4 and w_2 have no common neighbors in U. Since $d(w_1) + d(w_3) + 2(d(w_2) + d(w_4)) \geq 3\sigma_2(G)$, there exists a component of H_0'' mapped to some $U' \subset V(G)$ with $d_{U'}(w_1) + d_{U'}(w_3) + 2(d_{U'}(w_2) + d_{U'}(w_4)) > 4|U'|$. Note that $U' \neq U$, since $d_U(w_2) + d_U(w_4) \leq |U|$. Then $G[U \cup W \cup U']$ satisfies the conditions of Lemma 4, which proves this case.

CASE 5. $U = \{u_1, u_2, u_3, u_4\}$ and $G[U] = K_4$. If $G[W \cup U]$ contains two disjoint copies of K_4^-, then by Lemma 2, there are $u_4 \in U$ and $w_4 \in W$ such that (i) $N_{W-w_4}(u_1) = N_{W-w_4}(u_2) = N_{W-w_4}(u_3) = W - w_4$, and (ii) w_4 has at most one neighbor in U and $|E_G(W, U)| = 11$. Since $d(w_1) + d(w_3) + 2(d(w_2) + d(w_4)) \geq 3\sigma_2(G)$, there exists a component of H_0'' mapped to some $U' \subset V(G)$ with $d_{U'}(w_1) + d_{U'}(w_3) + 2(d_{U'}(w_2) + d_{U'}(w_4)) > 4|U'|$. By (ii), $U' \neq U$. Then $G[U \cup W \cup U']$ satisfies the conditions of Lemma 4, which finishes the proof.

5. Two reductions

In this section, we prove two lemmas that will help us later to find a special subgraph H in a graph satisfying (1).

Let the *microphone graph* M_1 be the 5-vertex graph such that a 4-vertex subgraph of M_1 is K_4 and the fifth vertex has exactly one neighbor in M_1.

Lemma 6. *Let H be an n-vertex graph whose components are isomorphic to graphs in $\mathcal{H} = \{K_1, K_2, C_3, K_4^-, C_5^+\}$. Let H_1 be the graph obtained from H by replacing a copy of C_5^+ with a copy of the microphone graph. If an n-vertex graph G satisfying (1) contains H_1, then it contains H, as well.*

Proof. Suppose not. Fix an embedding of H_1 into G. Suppose that the component M_1 of H_1 is embedded into the subset $A = \{a_1, a_2, a_3, a_4, a_5\}$ of $V(G)$ so that $G[A - a_5] = K_4$. Since G does not contain H, we may assume that the only neighbor of a_5 in A is a_4. For every $W \subseteq V(G)$, consider the expression $D(W) = 3d_W(a_5) + d_W(a_1) + d_W(a_2) + d_W(a_3)$. Since $D(V(G)) \geq 3\sigma_2$, we have $D(V(G) - A) \geq 3\sigma_2 - 2|E(G[A])| \geq (4n-3) - 14 > 4(n-5)$, and hence there exists a component of H_1 mapped to a set $U \subset V(G)$ with $D(U) > 4|U|$. Let $A_1 = \{a_1, a_2, a_3\}$.

CASE 1: $U = \{u\}$. By the choice of U, $D(U) \geq 5$. Then u is adjacent to a_5 and at least two vertices in A_1. Hence $G[A - a_5 + u]$ contains C_5^+.

CASE 2: $U = \{u_1, u_2\}$ and $G[U] = K_2$. By the choice of U, $D(U) \geq 9$. Then some vertex of U, say u_1, has at least two neighbors in A_1. If $a_5u_2 \in E(G)$, then $G[A - a_5 + u_1]$ contains C_5^+ and $G[u_2, a_5] = K_2$. If $a_5u_2 \notin E(G)$, then the only way to have $D(U) \geq 9$ is that u_1 is adjacent to all vertices in $A - a_4$ and u_2 is adjacent to all vertices in A_1. In this case, after switching the roles of u_2 and u_1, the previous argument works.

Observe that in order to have $D(U) > 4|U|$ for any U with $|U| \geq 3$, we need

$$d_U(a_5) \geq 2. \tag{11}$$

CASE 3: $U = \{u_1, u_2, u_3\}$ and $G[U] = K_3$. By the choice of U, $D(U) \geq 13$. Some vertex in U, say u_1, has at least two neighbors in A_1. If $a_5u_2, a_5u_3 \in E(G)$, then $G[A - a_5 + u_1]$ contains C_5^+ and $G[a_5, u_2, u_3] = K_3$. Hence we may assume that $N(a_5) \cap U = \{u_1, u_2\}$. Then by the above $d_{A_1}(u_3) \leq 1$. So, to have $D(U) \geq 13$, we need $N(u_2) \cap A_1 = N(u_1) \cap A_1 = A_1$ and $d_{A_1}(u_3) = 1$. Let $a \in A_1$ be the neighbor of u_3 in A_1. Then $G[a_4, a_5, u_1, u_3, a]$ contains C_5^+ and $G[A_1 - a + u_1]$ is a triangle, a contradiction.

CASE 4: $U = \{u_1, u_2, u_3, u_4\}$ and $G[U] \supset K_4^-$ with u_1u_3 as the only possible non-edge. By the choice of U, $D(U) \geq 17$. If $G[U + a_5]$ contains C_5^+, we are done. By (11), $G[C + a_5]$ does not contain C_5^+ only if $N(a_5) \cap U = \{u_2, u_4\}$ and $u_1u_3 \notin E(G)$. Therefore, there are at least 11 edges between A_1 and U, and we can find $a \in A_1$ and $u \in \{u_1, u_3\}$ such that $U - u \subseteq N(a)$ and $A_1 - a \in N(u)$. Then $G[U - u + a + a_5]$ contains C_5^+ and $G[A_1 - a_5 - a + u]$ contains K_4^-.

CASE 5: $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $G[U]$ contains cycle $(u_1, u_2, u_3, u_4, u_5)$ and edge u_2u_5. By the choice of U, $D(U) \geq 21$. We may assume that a_1 has the most neighbors in U among vertices in A_1. Since $D(U) \geq 21$,

$$d_U(a_5) + d_U(a_1) \geq 7. \quad (12)$$

Subcase 5.1: $d_U(a_1) = 5$. If some neighbor u of a_5 in U is adjacent to a_2 or a_3, then either of $G[A - a_1 + u]$ and $G[U - u + a_1]$ contains C_5^+, a contradiction. Otherwise, $e(A_1, U) \leq 15 - 2d_U(a_5)$ and hence $D(U) \leq 15 + d_U(a_5) \leq 20$, a contradiction again.

Subcase 5.2: $d_U(a_5) = 3$. Since $e(A_1, U) \leq 21 - 9 = 12$ and $d_U(a_1) \leq 4$, we have $d_U(a) = 4$, for each $a \in A_1$. Then a_5 has at least two common neighbors in U with a_1, say, u and u'. If $G[U - u + a_2] \supseteq C_5^+$ or $G[U - u' + a_2] \supseteq C_5^+$, then we are done, since in this case either of $G[A - a_2 + u]$ and $G[A - a_2 + u']$ also contains C_5^+. Otherwise, u and u' are the two vertices next on the cycle $(u_1, u_2, u_3, u_4, u_5)$ to the non-neighbor of a_2 in U, and the third neighbor of a_5 in U is the non-neighbor of a_1. By the symmetry between a_1, a_2 and a_3, we conclude that u and u' are adjacent to all vertices in A_1, and the third neighbor of a_5 is adjacent to none of them. Let the non-neighbors of a_5 in U be u_i and u_{i+1}. Then either of $G[U - a_3 + u_i]$ and $G[U - u_i + a_5 + a_3]$ contains C_5^+.

Subcase 5.3: $d_U(a_5) = 4$. Let u_i be the non-neighbor of a_5 in U. If some $u \in U - u_{i-1} - u_{i+1}$ has at least two neighbors in A_1, then either of $G[A - a_5 + u]$ and $G[U - u + a_5]$ contains C_5^+. Otherwise, to have $D(U) \geq 21$, we need $d_{A_1}(u_{i-1}) = d_{A_1}(u_{i+1}) = 3$ and $d_{A_1}(u_i) = d_{A_1}(u_{i-2}) = d_{A_1}(u_{i+2}) = 1$. Since $d_U(a_1) \leq 4$, no vertex in A_1 is a common neighbor of u_{i-2}, u_i, and u_{i+2}. By the symmetry between u_{i-2} and u_{i+2}, we may assume that for some distinct $a, a' \in A_1$, $u_{i-2}a, u_ia' \in E(G)$. Let a'' be the third vertex in A_1. Then either of $G[A - a - a' + u_{i-2} + u_{i+2}]$ and $G[U + a + a' - u_{i-2} - u_{i+2}]$ contains C_5^+.

Subcase 5.4: $d_U(a_5) = 5$. Since $D(U) \geq 21$, some $u \in U$ has at least 2 neighbors in A_1. Then either of $G[A - a_5 + u]$ and $G[U - u + a_5]$ contains C_5^+. □

The *T-graph* is the 5-vertex graph obtained from $K_{2,3}$ by adding an edge connecting the two vertices of degree 3. Equivalently, the *T-graph* is the 5-vertex graph obtained from K_5 by deleting the edges of a triangle. Sometimes, the T-graph is also called the *book with* 3 *pages*.

Lemma 7. *Let H be an n-vertex graph whose components are isomorphic to graphs in $\mathcal{H} = \{K_1, K_2, C_3, K_4^-, C_5^+\}$. Let H_2 be the graph obtained from H by replacing a copy of C_5^+ with a copy of the T-graph. If an n-vertex graph G satisfying (1) contains H_2, then it contains H, as well.*

Proof. Suppose not. Fix an embedding of H_2 into G. Suppose that the T-graph component of H_2 is mapped to a subset $A = \{a_1, a_2, a_3, a_4, a_5\}$ of $V(G)$ so that $d_A(a_4) = d_A(a_5) = 4$. Since G does not contain H, the set $A_1 = \{a_1, a_2, a_3\}$ is independent in G. For every $W \subseteq V(G)$, consider the expression $D(W) = d_W(a_1) + d_W(a_2) + d_W(a_3)$. Since $D(V(G)) \geq \frac{3}{2}\sigma_2$, we have $D(V(G) - A) \geq \frac{3}{2}\sigma_2 - 6 \geq (4n - 3)/2 - 6 > 2(n - 5)$, and hence there exists a component of H_2 mapped to a set $U \subset V(G)$ with $D(U) > 2|U|$. By symmetry, we may assume that

$$d_U(a_1) \geq d_U(a_2) \geq d_U(a_3). \quad (13)$$

CASE 1: $U = \{u\}$. By the choice of U, $D(U) \geq 3$. In particular, $a_1u, a_2u \in E(G)$. Then $G[A - a_3 + u]$ contains C_5^+.

CASE 2: $U = \{u_1, u_2\}$ and $G[U] = K_2$. By the choice of U, $D(U) \geq 5$. So, we may assume that among the edges connecting A_1 with U only a_3u_2 is missing. Then $G[a_3, u_1] = K_2$ and $G[A + u_2 - a_3]$ contains C_5^+.

CASE 3: $U = \{u_1, u_2, u_3\}$ and $G[U] = K_3$. By the choice of U, $D(U) \geq 7$. So, by (13), $d_U(a_1) = 3$ and $d_U(a_2) \geq 2$. Then $G[U \cup \{a_1, a_2\}]$ contains C_5^+ and $G[A - a_1 - a_2] = K_3$.

CASE 4: $U = \{u_1, u_2, u_3, u_4\}$ and $G[U] \supset K_4^-$. By the choice of U, $D(U) \geq 9$. Hence by (13), $d_U(a_1) \geq D(U)/3 \geq 3$. Then $G[U + a_1]$ contains C_5^+ and $G[A - a_1] = K_4^-$.

CASE 5: $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $G[U]$ contains cycle $(u_1, u_2, u_3, u_4, u_5)$ and edge u_2u_5. By the choice of U, $D(U) \geq 11$. By (13), $d_U(a_1) \geq 4$.

Subcase 5.1: $d_U(a_1) = 5$. Since $d_U(a_2) + d_U(a_3) \geq 6$, some $u \in U$ is adjacent to both of them. Then either of $G[A - a_1 + u]$ and $G[U - u + a_1]$ contains C_5^+.

Subcase 5.2: $d_U(a_1) = 4$. Then $d_U(a_2) = 4$ and $d_U(a_3) \geq 3$ also. Let u_i be the only non-neighbor of a_1 in U. If any $u \in U - u_{i-1} - u_{i+1}$ is adjacent to both a_2 and a_3, then again either of $G[A - a_1 + u]$ and $G[U - u + a_1]$ contains C_5^+. Otherwise, the only possibility to have $D(U) \geq 11$ is that $d_{A_1}(u_{i-1}) = d_{A_1}(u_{i+1}) = 3$, $d_{A_1}(u_{i-2}) = d_{A_1}(u_{i+2}) = 2$, and $d_{A_1}(u_i) = 1$.

Let $a_j = a_1$ when $d_{A_1}(u_1) = 3$, and let a_j be the only non-neighbor of u_1 in A_1 when $d_{A_1}(u_1) = 2$. So, if $d_{A_1}(u_1) \geq 2$, then $G[A - a_j + u_1]$ contains a C_5^+. Furthermore, since $d_{U-u_1}(a_j) \geq 3$ and $u_2u_5 \in E(G)$, $G[U + a_j - u_1]$ also contains C_5^+. Thus, the only remaining possibility is that $i = 1$. By the symmetry between a_1 and a_2, the only non-neighbor of a_2 in U is also u_1. Hence $N_U(a_3) = \{u_5, u_1, u_2\}$. Then $G[A - a_3 + u_4]$ contains C_5^+ and $G[U - u_4 + a_3]$ is the microphone graph. This means that G contains the graph H_1 obtained from H by replacing a copy of C_5^+ by a copy of the microphone graph. Hence by Lemma 6, G contains H, a contradiction. □

6. Proof of Theorem 3

Similarly to Section 4, let $\mathcal{H}_n$ be the class of n-vertex graphs whose every component is either K_1, or K_2, or K_3, or K_4^-, or C_5^+. Let G satisfy the conditions of the theorem. Suppose, for a contradiction, that G does not contain some graph in $\mathcal{H}_n$. Among such 'bad' graphs in $\mathcal{H}_n$ choose a graph H_0 with fewest components that are C_5^+. By Theorem 4, H_0 has a C_5^+-component. Let H_0' be obtained from H_0 by replacing a C_5^+-component with K_4^- and an isolated vertex. By the minimality of H_0, there exists an embedding of H_0' in G. Among embeddings of H_0' in G, choose and fix one such that

(*) *it has the largest total number of edges in subgraphs of G induced by the components of H_0'.*

Suppose that the isolated vertex of H_0' is mapped to a vertex $v \in V(G)$ and a K_4^--component of H_0' is mapped to a set $W = \{w_1, w_2, w_3, w_4\} \subset V(G)$, where only w_1w_3 can be a non-edge of $G[W]$. Since $G[W + v]$ does not contain C_5^+, only the three cases below are possible up to symmetry.

CASE 1. $N(v) \cap W \subseteq \{w_1\}$. For every $Y \subseteq V(G)$, consider the expression $D(Y) = 3d_Y(v) + d_Y(w_2) + d_Y(w_3) + d_Y(w_4)$. Since $D(V(G)) \geq 3\sigma_2$, we have $D(V(G) - W - v) \geq 3\sigma_2 - (3 + 3 + 3 + 3) \geq (4n - 3) - 12 > 4(n - 5)$, and hence there exists a component of H_0' mapped to a set $U \subset V(G)$ with $D(U) > 4|U|$. Denote $W_1 = W - w_1$.

Case 1.1: $U = \{u\}$. By the choice of U, $D(U) \geq 5$. Then u is adjacent to v and to at least two vertices in W_1. Hence $G[W + u]$ contains either C_5^+ or the T-graph. By Lemma 7, this contradicts the choice of G.

Case 1.2: $U = \{u_1, u_2\}$ and $G[U] = K_2$. By the choice of U, $D(U) \geq 9$. Then some vertex of U, say u_1, has at least two neighbors in W_1. If $vu_2 \in E(G)$, then $G[u_2, v] = K_2$ and $G[W + u_1]$ contains either C_5^+ or the T-graph. If $vu_2 \notin E(G)$, then the only way to have $D(U) \geq 9$, is that both u_1 and u_2 are adjacent to all vertices in W_1 and $u_1v \in E(G)$. But then after switching the roles of u_2 and u_1, the previous argument works.

Observe that in order to have $D(U) > 4|U|$ for a U with $|U| \geq 3$, we need

$$d_U(v) \geq 2. \tag{14}$$

Case 1.3: $U = \{u_1, u_2, u_3\}$ and $G[U] = K_3$. By the choice of U, $D(U) \geq 13$. Some vertex in U, say u_1, has at least two neighbors in W_1. If $vu_2, vu_3 \in E(G)$, then $G[v, u_2, u_3]$ is a triangle and $G[W + u_1]$ contains either C_5^+ or the T-graph. Hence we may assume that $vu_3 \notin E(G)$ and, by (14), $N(w) \cap U = \{u_1, u_2\}$. Then by the above argument, $d_{W_1}(u_3) \leq 1$. So, to have $D(U) \geq 13$, we need $N(u_2) \cap W_1 = N(u_1) \cap W_1 = W_1$ and $d_{W_1}(u_3) = 1$. In this case, $G[U + v + w_3]$ contains the T-graph and $G[W - w_3] = K_3$.

Case 1.4: $U = \{u_1, u_2, u_3, u_4\}$ and $G[U] \supset K_4^-$ with u_1u_3 as the only possible non-edge. By (14), $G[U + v]$ contains either C_5^+, or the T-graph.

Case 1.5: $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $G[U]$ contains cycle $(u_1, u_2, u_3, u_4, u_5)$ and edge u_2u_5. By the choice of U, $D(U) \geq 21$.

Subcase 1.5.1: $d_U(v) = 5$. Since $D(U) \geq 21$, some $u \in U$ has at least two neighbors in $W - w_1$. Then $G[U - u + v]$ contains C_5^+ and $G[W + u]$ contains either C_5^+, or the T-graph.

Subcase 1.5.2: $d_U(v) = 4$. Let u_i be the only non-neighbor of v in U. If $d_U(u_i) < 4$, then $G[U - u_i + v]$ has more edges than $G[U]$, a contradiction to (*). So, $d_U(u_i) = 4$. Hence for each j, $G[U - u_j + v]$ contains C_5^+. If follows that we are done if for some j, u_j has at least two neighbors in W. Otherwise, $D(U) \leq 3 \cdot 4 + 5 \cdot 1 = 17$, a contradiction.

Subcase 1.5.3: $d_U(v) = 3$. Then $e(W_1, U) \geq 21 - 9 = 12$. If for some $u \in U$, $G[U - u + v]$ contains C_5^+, then u has at most one neighbor in W, because otherwise $G[W + u]$ contains either C_5^+ or the T-graph. Since at most 3 edges connecting W_1 with U are missing, it yields:

(**) *There is at most one $u \in U$ such that $G[U - u + v]$ contains C_5^+.*

If the non-neighbors of v in U are not consecutive on the cycle $(u_1, \ldots, u_5)$ (in which case they are u_{i-1} and u_{i+1} for some $i \in \{1, 2, 3, 4, 5\}$), then both $G[U - u_{i-1} + v]$ and $G[U - u_{i+1} + v]$ contain C_5^+, a contradiction to (**). So we assume below that the neighbors of v in U are u_{i-1}, u_i, and u_{i+1}. Recall that $u_2u_5 \in E(G)$. Up to a symmetry, there are three possibilities: $i = 3$, $i = 2$ and $i = 1$. If $i = 3$, then either of $G[U - u_1 + v]$ and $G[U - u_3 + v]$ contains C_5^+, a contradiction to (**), again. If $i = 2$, then either of $G[U - u_1 + v]$ and $G[U - u_4 + v]$ contains C_5^+. Thus, the last possibility is that $i = 1$. In this case, u_1 has at most one neighbor in W and hence some $u \in \{u_3, u_4\}$ is adjacent to all vertices in W_1. Then $G[W + u]$ contains C_5^+, and $G[U - u + v]$ contains the microphone graph. This contradicts Lemma 6.

Subcase 1.5.4: $d_U(v) = 2$. Then $e(W - w_1, U) \geq 21 - 6 = 15$. It follows that all edges connecting U with $W - w_1$ are present in G. We may assume that the neighbors of v in U are u_i and either u_{i+1} or u_{i+2}. Then either of $G[u_i, u_{i+1}, u_{i+2}, v, w_3]$ and $G[W - w_3 + u_{i-1} + u_{i-2}]$ contains C_5^+.

CASE 2. $N(v) \cap W = \{w_2, w_4\}$. Then v and W form the T-graph, and we are done by Lemma 7.

CASE 3. $N(v) \cap W = \{w_4\}$. For every $Y \subseteq V(G)$, let $D(Y) = 3d_Y(v) + d_Y(w_1) + d_Y(w_2) + d_Y(w_3)$. Since $D(V(G)) \geq 3\sigma_2$, we have $D(V(G) - W - v) \geq 3\sigma_2 - (3 + 3 + 3 + 3) \geq (4n - 3) - 12 > 4(n - 5)$, and hence there exists a component of H_0' mapped to a set $U \subset V(G)$ with $D(U) > 4|U|$. Let $W_4 = W - w_4$.

Proofs of the Cases 3.1 (when $|U| = 1$), 3.2 (when $|U| = 2$) and 3.4 (when $|U| = 4$) are exact repetitions of proofs of the Cases 1.1, 1.2, and 1.4, respectively. Also, (14) holds if $|U| \geq 3$ for the same reasons as in Case 1.

Case 3.3: $U = \{u_1, u_2, u_3\}$ and $G[U] = K_3$. By the choice of U, $D(U) \geq 13$. If some $w \in \{w_1, w_3\}$ has at least two neighbors in U, then $G[W - w] = K_3$ and, by (14), $G[U + v + w]$ contains either C_5^+ or the T-graph. Otherwise, $e(W_4, U) \leq 5$ and to have $D(U) \geq 13$, we need $d_U(v) = 3$. In this case, we still have $e(W_4, U) \geq 4$ and hence some $w \in \{w_1, w_3\}$ has a neighbor in U. Then $G[U + v + w]$ contains the microphone graph and again $G[W - w] = K_3$.

Case 3.5: $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $G[U]$ contains cycle $(u_1, u_2, u_3, u_4, u_5)$ and edge u_2u_5. By the choice of U, $D(U) \geq 21$. The proofs of the subcases when $d_U(v)$ equals 5, 4, and 2 word-by-word repeat the proofs of the subcases 1.5.1, 1.5.2, and 1.5.4, respectively. So, we need to handle only the case $d_U(v) = 3$.

Since $D(U) \geq 21$, we have $e(W_4, U) \geq 12$. Then some $x \in \{w_1, w_3\}$ has at least four neighbors in U, and some $x' \in W_4 - x$ (also having at least four neighbors in U) has at least two common neighbors (say u and u') with v in U. Since $vw_4 \in E(G)$, $G[W - x + v + u]$ (and $G[W - x + v + u']$) contains either C_5^+ or the T-graph. Hence we are done if $G[U - u + x]$ or $G[U - u' + x]$ contains C_5^+. If neither of $G[U - u + x]$ and $G[U - u' + x]$ contains C_5^+, then $d_U(x) = 4$ and $d_U(x') = 4$. Furthermore, if u_i is the non-neighbor of x in U, then the common neighbors of v and x' in U are only u_{i-1} and u_{i+1}. It follows that $N_U(w_j) = U - u_i$ for $j = 1, 2, 3$ and $N_U(v) = \{u_{i-1}, u_i, u_{i+1}\}$. Then either of $G[u_{i-1}, u_i, u_{i+1}, v, w_1]$ and $G[W - w_1 + u_{i-2} + u_{i+2}]$ contains C_5^+.

So, all cases are considered and the theorem is proved.

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References

[1] M. Aigner, S. Brandt, Embedding arbitrary graphs of maximum degree two, J. London Math. Soc. (2) 28 (1993) 39–51.
[2] N. Alon, E. Fischer, 2-factors in dense graphs, Discrete Math. 152 (1996) 13–23.
[3] B. Bollobás, S.E. Eldridge, Packing of graphs and applications to computational complexity, J. Comb. Theory Ser. B 25 (1978) 105–124.
[4] B. Bollobás, A. Kostochka, K. Nakprasit, Packing d-degenerate graphs, J. Combin. Theory Ser. B (2007) doi:10.1016/j.jctb.2007.05.002.
[5] P.A. Catlin, Subgraphs of graphs. I, Discrete Math. 10 (1974) 225–233.
[6] P.A. Catlin, Embedding subgraphs and coloring graphs under extremal degree conditions, Ph. D. Thesis, Ohio State Univ., Columbus, 1976.
[7] K. Corrádi, A. Hajnal, On the maximum number of independent circuits in a graph, Acta Math. Acad. Sci. Hung. 14 (1963) 423–439.
[8] B. Csaba, A. Shokoufandeh, E. Szemerédi, Proof of a conjecture of Bollobás and Eldridge for graphs of maximum degree three, Combinatorica 23 (2003) 35–72.
[9] G. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2 (1952) 69–81.
[10] H. Enomoto, On the existence of disjoint cycles in a graph, Combinatorica 18 (1998) 487–492.
[11] G. Fan, H.A. Kierstead, Hamiltonian square paths, J. Combin. Theory Ser. B 67 (1996) 167–182.
[12] A. Hajnal, E. Szemerédi, Proof of conjecture of Erdős, in: P. Erdós, A. Rényi, V.T. Sós (Eds.), Combinatorial Theory and its Applications, vol. II, North-Holland, 1970, pp. 601–623.
[13] P. Justesen, On independent circuits in finite graphs and a conjecture of Erdós and Posa, Ann. Discrete Math. 41 (1989) 299–306.
[14] H. Kaul, A. Kostochka, Extremal graphs for a graph packing theorem of Sauer and Spencer, Combin. Probab. Comput. 16 (3) (2007) 409–416.
[15] A. Kostochka, G. Yu, Ore-type graph packing problems, Combin. Probab. Comput. 16 (2007) 167–169.
[16] A. Kostochka, G. Yu, An Ore-type analogue of the Sauer-Spencer Theorem, Graphs Combin. 23 (2007) 419–424.
[17] O. Ore, Note on hamilton circuits, Amer. Math. Monthly 67 (1960) 55.
[18] N. Sauer, J. Spencer, Edge disjoint placement of graphs, J. Combin. Theory Ser. B 25 (1978) 295–302.
[19] H. Wang, On the maximum number of independent cycles in a graph, Discrete Math. 205 (1999) 183–190.
[20] D.B. West, Introduction to Graph Theory, 2nd ed., Prentice Hall, Upper Saddle River, 2001.